

BLOWUP ALGEBRAS OF RATIONAL NORMAL SCROLLS

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ABSTRACT. We investigate the algebraic relations among the minors of a $2 \times c$ matrix with d catalecticant blocks, which define a d -fold rational normal scroll $\mathcal{S} \subseteq \mathbb{P}^{c+d-1}$. We determine the equations of the blowup of \mathbb{P}^{c+d-1} along \mathcal{S} and of the variety parametrized by these minors, or equivalently the relations of the Rees ring and special fiber ring of the homogeneous ideal of \mathcal{S} . Moreover, we prove that these blowup algebras are defined by squarefree Gröbner bases of quadrics and are Koszul.

INTRODUCTION

Let $R = \mathbb{K}[x_0, \dots, x_n]$ be a polynomial ring over a field and $I \subseteq R$ an ideal generated by forms $\mathbf{f} = \{f_0, \dots, f_s\}$ of the same degree. The *Rees ring* of I is $\mathcal{R}(I) = R[It] \subseteq R[t]$ and its *special fiber ring* is $\mathcal{R}(I) \otimes_R \mathbb{K} \cong \mathbb{K}[\mathbf{f}] \subseteq R$. They are known collectively as *blowup algebras* of I ; in fact, $\mathcal{R}(I)$ is the bihomogeneous coordinate ring of the blowup of \mathbb{P}^n along the subscheme defined by I . In general, $\text{BiProj}(\mathcal{R}(I))$ is the graph of the rational map $\mathbb{P}^n \xrightarrow{[f_0:\dots:f_s]} \mathbb{P}^s$, whereas $\text{Proj}(\mathbb{K}[\mathbf{f}])$ is the variety parametrized by the forms \mathbf{f} . Interesting information about rational maps can be obtained from these algebras [15, 22, 31, 33]. Blowup algebras are important objects in algebraic geometry as well as in the applied context of geometric modeling [7, 12]. From a more algebraic point of view, $\mathcal{R}(I)$ and $\mathbb{K}[\mathbf{f}]$ encode homological information on the powers of I [3, 5, 9, 13, 17, 18, 29].

A central problem about blowup algebras is the determination of their defining ideals. This amounts to finding the implicit equations of varieties parametrized by given homogeneous polynomials, and for this reason is classically known as the *implicitization problem*. Despite being well-studied in the last few decades, it remains a challenging problem, and the solution is known for very few classes, including linearly presented ideals I that are Cohen-Macaulay of codimension two [24] or Gorenstein of codimension three [23], cf. also [36]. It is even open for three-generated ideals $I \subseteq \mathbb{K}[x_0, x_1]$ [7].

Of significant interest are the defining relations of subalgebras $\mathbb{K}[\mathbf{f}] \subseteq R$ when the forms \mathbf{f} are minors (or pfaffians) of generic, symmetric, skew-symmetric, or catalecticant matrices; even in these classical situations though, not much is known. In the notable case of maximal minors of a generic matrix, $\mathbb{K}[\mathbf{f}]$ is the coordinate ring of a Grassmann variety and is defined by the well known Plücker relations. However, the relations among non-maximal minors of a generic matrix are still unknown [4]. For symmetric matrices,

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a set of generators for the defining ideal of $\mathbb{K}[\mathbf{f}]$ when \mathbf{f} consists of all principal minors is conjectured in [20], and established in [26] up to radical. Very little is known in the cases of symmetric and skew-symmetric matrices when one considers principal minors or pfaffians of fixed size [14].

We focus on rational normal scrolls, which are determinantal varieties of matrices obtained by concatenating catalecticant blocks. Let n_1, \dots, n_d be positive integers and $c = \sum n_i$. Choose rational normal curves $\mathcal{C}_i \subseteq \mathbb{P}^{c+d-1}$ of degree n_i with complementary linear spans, and isomorphisms $\varphi_i : \mathbb{P}^1 \rightarrow \mathcal{C}_i$ for each $i = 1, \dots, d$. The corresponding rational normal scroll is the variety

$$\mathcal{S}_{n_1, \dots, n_d} = \bigcup_{p \in \mathbb{P}^1} \overline{\varphi_1(p), \varphi_2(p), \dots, \varphi_d(p)} \subseteq \mathbb{P}^{c+d-1}$$

and is uniquely determined by n_1, \dots, n_d up to projective equivalence. Choosing suitable coordinates, the homogeneous ideal of $\mathcal{S}_{n_1, \dots, n_d}$ is generated by the minors of the matrix

$$\left(\begin{array}{cccc|cccc| \cdots |} x_{1,0} & x_{1,1} & \cdots & x_{1,n_1-1} & x_{2,0} & x_{2,1} & \cdots & x_{2,n_2-1} & \cdots & x_{d,0} & x_{d,1} & \cdots & x_{d,n_d-1} \\ x_{1,1} & x_{1,2} & \cdots & x_{1,n_1} & x_{2,1} & x_{2,2} & \cdots & x_{2,n_2} & \cdots & x_{d,1} & x_{d,2} & \cdots & x_{d,n_d} \end{array} \right).$$

Scrolls are prominent varieties in algebraic geometry, both in classical results and more recent developments, cf. for example [1, 2, 8, 16, 21]. They are varieties of minimal degree, and play a central role in the celebrated classification of Del Pezzo and Bertini.

A complete solution to the implicitization problem was given by Conca, Herzog, and Valla [11] for *balanced* scrolls $\mathcal{S}_{n_1, \dots, n_d}$, i.e. when $|n_i - n_j| \leq 1$ for all i, j . Under the balancedness assumption, I is also the ideal of minors of a single $2 \times c$ “generalized catalecticant” matrix. The authors exploited the combinatorics of such matrices to find Sagbi bases (i.e. deformation to toric rings) of $\mathcal{R}(I)$ and $\mathbb{K}[\mathbf{f}]$, and show that they are defined by quadratic squarefree Gröbner bases and are thus Koszul algebras. However, their technique does not extend to the general case [11, Example 3.9].

By a result of Blum [3], the Koszul property of $\mathcal{R}(I)$ implies that I has linear powers, in other words the Castelnuovo-Mumford regularity of the powers of I is the smallest possible. Recently, Bruns, Conca, and Varbaro [5] proved that all rational normal scrolls have linear powers. The method used is different from the one of [11] though, and (the degrees of) the defining equations, the Koszul property, and the existence of quadratic Gröbner bases for the blowup algebras of arbitrary scrolls remained unknown.

In this paper we solve the implicitization problem for arbitrary rational normal scrolls, cf. the Main Theorem in Section 1. In other words, we determine the equations of the blowup of \mathbb{P}^{c+d-1} along $\mathcal{S}_{n_1, \dots, n_d}$ and of the special fiber in their natural embeddings in $\mathbb{P}^{c+d-1} \times \mathbb{P}^{\binom{c-1}{2}-1}$ and $\mathbb{P}^{\binom{c-1}{2}-1}$ respectively. We exhibit quadratic relations, arising from the vanishing of certain 4×4 determinants, which, together with the classical Plücker relations and the syzygies of I , suffice to generate the ideals of relations of the blowup algebras. More importantly, we show that these quadrics form squarefree Gröbner bases for the ideals of relations of $\mathcal{R}(I)$ and $\mathbb{K}[\mathbf{f}]$ with respect to suitable term orders, implying the Koszul property for the two rings. It follows that these statements hold for all projective varieties of minimal degree, with the single exception of the Veronese surface in \mathbb{P}^5 , cf. Remark 5.18.

A crucial tool in this paper is the introduction of an unconventional term order. In fact, it seems that the defining equations of $\mathcal{R}(I)$ and $\mathbb{K}[\mathbf{f}]$ fail to form Gröbner bases for most reasonable term orders, including any generalization of the order used in [11] in the balanced case. To this end, we need to work with an alternative presentation of the ideal I , obtained by a convenient rearrangement of the columns of the matrix of the scroll. This presentation is described in Section 2. We construct the term order in Section 3 by refining a suitable lexicographic order with a single grading and then with a multigrading.

Through the term order we attach a flag simplicial complex to the special fiber. A combinatorial analysis carried out in Section 4 allows to deduce that this complex has the expected invariants. Since the Hilbert function of $\mathbb{K}[\mathbf{f}]$ is uniquely determined by the dimension and the codimension of the scroll [5], we can conclude that our set of defining equations of $\mathbb{K}[\mathbf{f}]$ is a generating set and a Gröbner basis. A pivotal step is the enumeration of the facets of the complex, which obey the same combinatorial patterns as the Catalan trapezoids, a generalization of the well-known Catalan numbers.

Combining the result on $\mathbb{K}[\mathbf{f}]$ with the fact that the ideal I is of fiber type [5], we deduce the defining equations of $\mathcal{R}(I)$. Section 5 is devoted to proving that these equations form a Gröbner basis, and we proceed by applying Buchberger's criterion. We employ a mixed strategy to show that all the S-pairs reduce to 0. We first identify some smaller Gröbner bases and, by reducing to a finite number of ideals, we avoid many explicit S-pair reductions. For the remaining S-pairs, we produce equations of reduction from some syzygies of the defining ideal of $\mathcal{R}(I)$, which have a determinantal structure.

The computer algebra software Macaulay2 [19] provided valuable help during the development of this paper.

1. RELATIONS AMONG MINORS

In this section we use the standard presentation of the ideal of a rational normal scroll to describe the equations of its blowup algebras in detail, and state our main results. We refer to [6, 25] for generalities on determinantal ideals.

From now on we assume, without loss of generality, that $n_1 \leq n_2 \leq \dots \leq n_d$. For each $i = 1, \dots, d$ let $X_i = \{x_{i,0}, \dots, x_{i,n_i}\}$ be disjoint sets of variables and $R = \mathbb{K}[x_{i,j}]$ a polynomial ring in the union of the X_i over an arbitrary field \mathbb{K} . Consider the matrix

$$\mathbf{X} = \left(\begin{array}{cccc|cccc|cccc} x_{1,0} & x_{1,1} & \cdots & x_{1,n_1-1} & x_{2,0} & x_{2,1} & \cdots & x_{2,n_2-1} & \cdots & x_{d,0} & x_{d,1} & \cdots & x_{d,n_d-1} \\ x_{1,1} & x_{1,2} & \cdots & x_{1,n_1} & x_{2,1} & x_{2,2} & \cdots & x_{2,n_2} & \cdots & x_{d,1} & x_{d,2} & \cdots & x_{d,n_d} \end{array} \right).$$

and denote by $\xi_{i,j}$ its (i, j) -entry. The homogeneous ideal of $\mathcal{S}_{n_1, \dots, n_d}$ is $I = (\mathbf{f}) \subseteq R$ where $\mathbf{f} = \{f_{\alpha, \beta} \mid 1 \leq \alpha < \beta \leq c\}$, $f_{\alpha, \beta} = \xi_{1, \alpha} \xi_{2, \beta} - \xi_{1, \beta} \xi_{2, \alpha}$, and $c = \sum n_i$.

Let t be a new variable. The Rees ring $\mathcal{R}(I) = R[It] \subseteq R[t]$ is bigraded by letting $\deg x_{i,j} = (1, 0)$ and $\deg t = (-2, 1)$. In this way $\mathcal{R}(I)$ is a standard bigraded \mathbb{K} -algebra in the sense that it is generated in bidegrees $(1, 0)$ and $(0, 1)$. The special fiber ring $\mathcal{R}(I) \otimes_R \mathbb{K} \cong \mathbb{K}[\mathbf{f}] \subseteq R$ can be identified with the subring of $\mathcal{R}(I)$ concentrated in bidegrees $(0, *)$, and is a standard graded \mathbb{K} -algebra. Introduce new variables $\{Y_{\alpha, \beta} \mid 1 \leq \alpha < \beta \leq c\}$

and define a bihomogeneous presentation

$$\Psi_{\mathcal{R}} : R[Y_{\alpha,\beta}] \twoheadrightarrow \mathcal{R}(I), \quad Y_{\alpha,\beta} \mapsto tf_{\alpha,\beta}, \quad x_{i,j} \mapsto x_{i,j},$$

and the induced homogeneous presentation

$$\Psi_{\mathcal{F}} : \mathbb{K}[Y_{\alpha,\beta}] \twoheadrightarrow \mathbb{K}[\mathbf{f}], \quad Y_{\alpha,\beta} \mapsto f_{\alpha,\beta}.$$

The ideal $\ker \Psi_{\mathcal{R}}$ is bigraded, while $\ker \Psi_{\mathcal{F}}$ is the graded ideal generated by the elements of $\ker \Psi_{\mathcal{R}}$ of bidegree $(0, *)$, in other words $\ker \Psi_{\mathcal{F}} = \ker \Psi_{\mathcal{R}} \cap \mathbb{K}[Y_{\alpha,\beta}]$.

Notation 1.1. For a positive integer n we set $[n] = \{1, \dots, n\}$. Given $\Gamma \subseteq [c]$ and $k \in \mathbb{N}$ we denote by $\binom{\Gamma}{k}$ the set of strictly increasing k -tuples of elements of Γ .

Some relations of the Rees ring derive from the first syzygies of I . The resolution of the ideal of maximal minors of a matrix is well-understood thanks to the Eagon-Nortcott complex. In our case, we can describe the first syzygies as follows: for any $(\alpha, \beta, \gamma) \in \binom{[c]}{3}$ the following determinants vanish

$$\begin{vmatrix} \xi_{1,\alpha} & \xi_{1,\beta} & \xi_{1,\gamma} \\ \xi_{1,\alpha} & \xi_{1,\beta} & \xi_{1,\gamma} \\ \xi_{2,\alpha} & \xi_{2,\beta} & \xi_{2,\gamma} \end{vmatrix} = \begin{vmatrix} \xi_{1,\alpha} & \xi_{1,\beta} & \xi_{1,\gamma} \\ \xi_{2,\alpha} & \xi_{2,\beta} & \xi_{2,\gamma} \\ \xi_{2,\alpha} & \xi_{2,\beta} & \xi_{2,\gamma} \end{vmatrix} = 0.$$

Expanding the determinants along the first and third row respectively gives rise to

$$(1) \quad \xi_{1,\alpha}Y_{\beta,\gamma} - \xi_{1,\beta}Y_{\alpha,\gamma} + \xi_{1,\gamma}Y_{\alpha,\beta}, \quad \xi_{2,\alpha}Y_{\beta,\gamma} - \xi_{2,\beta}Y_{\alpha,\gamma} + \xi_{2,\gamma}Y_{\alpha,\beta} \in \ker \Psi_{\mathcal{R}}.$$

The Plücker relations are another well-known set of relations among maximal minors. For a $2 \times c$ matrix they take the form

$$(2) \quad Y_{\alpha,\beta}Y_{\gamma,\delta} - Y_{\alpha,\gamma}Y_{\beta,\delta} + Y_{\alpha,\delta}Y_{\beta,\gamma} \in \ker \Psi_{\mathcal{R}}, \quad (\alpha, \beta, \gamma, \delta) \in \binom{[c]}{4}.$$

The structure of the matrix \mathbf{X} yields other relations as well. For any four columns avoiding the terminal column of each catalecticant block, that is, for any $(\alpha, \beta, \gamma, \delta) \in \binom{\Gamma}{4}$ with $\Gamma = [c] \setminus \{n_1, n_1 + n_2, \dots, \sum n_i\}$, the following determinant vanishes

$$\begin{vmatrix} \xi_{1,\alpha} & \xi_{1,\beta} & \xi_{1,\gamma} & \xi_{1,\delta} \\ \xi_{2,\alpha} & \xi_{2,\beta} & \xi_{2,\gamma} & \xi_{2,\delta} \\ \xi_{1,\alpha+1} & \xi_{1,\beta+1} & \xi_{1,\gamma+1} & \xi_{1,\delta+1} \\ \xi_{2,\alpha+1} & \xi_{2,\beta+1} & \xi_{2,\gamma+1} & \xi_{2,\delta+1} \end{vmatrix} = 0$$

and expanding the determinant along the first two rows we get the element of $\ker \Psi_{\mathcal{R}}$

$$(3) \quad Y_{\alpha,\beta}Y_{\gamma+1,\delta+1} - Y_{\alpha,\gamma}Y_{\beta+1,\delta+1} + Y_{\alpha,\delta}Y_{\beta+1,\gamma+1} + Y_{\beta,\gamma}Y_{\alpha+1,\delta+1} - Y_{\beta,\delta}Y_{\alpha+1,\gamma+1} + Y_{\gamma,\delta}Y_{\alpha+1,\beta+1}.$$

We will prove that these polynomials suffice to generate the ideals $\ker \Psi_{\mathcal{R}}$ and $\ker \Psi_{\mathcal{F}}$.

Main Theorem. *The defining ideal of $\mathbb{K}[\mathbf{f}]$ is minimally generated by the polynomials (2) and (3), whereas the defining ideal of $\mathcal{R}(I)$ is minimally generated by the polynomials (1), (2), and (3). Moreover, these generating sets are Gröbner bases with respect to suitable term orders.*

The rest of the paper is devoted to the proof of this result, cf. Theorems 4.8, 5.1, and 5.15. In order to prove the theorem though, it will be necessary to employ a different, less conventional, presentation of the ideal I .

2. AN ALTERNATIVE PRESENTATION

We introduce a new matrix \mathbf{M} obtained by rearranging the columns of \mathbf{X} with a two-step process. We start with the first column of the i -th catalecticant block of \mathbf{X} for each i increasingly in i , then the second column, and so on until we have used all columns except the last one for each block. If a block runs out of columns before the others we simply skip it. Thus, if $i(\ell)$ is the first i such that $n_i \geq \ell$, the first $c - d$ columns of \mathbf{M} are

$$\begin{array}{cccccccccccc} x_{i(2),0} & x_{i(2)+1,0} & \cdots & x_{d,0} & x_{i(3),1} & x_{i(3)+1,1} & \cdots & x_{d,1} & x_{i(4),2} & \cdots & \cdots & n_{d,n_d-2} \\ x_{i(2),1} & x_{i(2)+1,1} & \cdots & x_{d,1} & x_{i(3),2} & x_{i(3)+1,2} & \cdots & x_{d,2} & x_{i(4),3} & \cdots & \cdots & n_{d,n_d-1} \end{array}.$$

The last d columns of \mathbf{M} consist of the last column of the i -th block of \mathbf{X} for each i , this time ordered decreasingly in i :

$$\begin{array}{cccccc} x_{d,n_d-1} & x_{d-1,n_{d-1}-1} & \cdots & x_{2,n_2-1} & x_{1,n_1-1} & \\ x_{d,n_d} & x_{d-1,n_{d-1}} & \cdots & x_{2,n_2} & x_{1,n_1} & \end{array}.$$

Examples 2.1. We illustrate the construction with several examples. Let $\mathbf{n} = (n_1, \dots, n_d)$.

$$\begin{aligned} \mathbf{n} = (4, 4) \quad \mathbf{M} &= \begin{pmatrix} x_{1,0} & x_{2,0} & x_{1,1} & x_{2,1} & x_{1,2} & x_{2,2} & x_{2,3} & x_{1,3} \\ x_{1,1} & x_{2,1} & x_{1,2} & x_{2,2} & x_{1,3} & x_{2,3} & x_{2,4} & x_{1,4} \end{pmatrix}, \\ \mathbf{n} = (3, 3, 4) \quad \mathbf{M} &= \begin{pmatrix} x_{1,0} & x_{2,0} & x_{3,0} & x_{1,1} & x_{2,1} & x_{3,1} & x_{3,2} & x_{3,3} & x_{2,2} & x_{1,2} \\ x_{1,1} & x_{2,1} & x_{3,1} & x_{1,2} & x_{2,2} & x_{3,2} & x_{3,3} & x_{3,4} & x_{2,3} & x_{1,3} \end{pmatrix}, \\ \mathbf{n} = (1, 1, 2, 3, 4) \quad \mathbf{M} &= \begin{pmatrix} x_{3,0} & x_{4,0} & x_{5,0} & x_{4,1} & x_{5,1} & x_{5,2} & x_{5,3} & x_{4,2} & x_{3,1} & x_{2,0} & x_{1,0} \\ x_{3,1} & x_{4,1} & x_{5,1} & x_{4,2} & x_{5,2} & x_{5,3} & x_{5,4} & x_{4,3} & x_{3,2} & x_{2,1} & x_{1,1} \end{pmatrix}, \\ \mathbf{n} = (1, 1, \dots, 1) \quad \mathbf{M} &= \begin{pmatrix} x_{d,0} & x_{d-1,0} & x_{d-2,0} & \cdots & x_{2,0} & x_{1,0} \\ x_{d,1} & x_{d-1,1} & x_{d-2,1} & \cdots & x_{2,1} & x_{1,1} \end{pmatrix}, \\ \mathbf{n} = (n_1) \quad \mathbf{M} &= \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & \cdots & x_{1,n_1-1} \\ x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n_1} \end{pmatrix}. \end{aligned}$$

Denote by $\mu_{i,j}$ the (i, j) -entry of \mathbf{M} and by $g_{\alpha,\beta} = \mu_{1,\alpha}\mu_{2,\beta} - \mu_{1,\beta}\mu_{2,\alpha}$ its 2×2 minors, with $\alpha < \beta$. Thus $I = (\mathbf{g}) \subseteq R$ where $\mathbf{g} = \{g_{\alpha,\beta} \mid (\alpha, \beta) \in \binom{[c]}{2}\}$. We let $\{T_{\alpha,\beta} \mid (\alpha, \beta) \in \binom{[c]}{2}\}$ be new variables and define (bi)homogeneous presentations for the Rees ring and the special fiber ring of I

$$\begin{aligned} \Pi_{\mathcal{R}} : S_{\mathcal{R}} := R[T_{\alpha,\beta}] &\twoheadrightarrow \mathcal{R}(I), & T_{\alpha,\beta} &\mapsto tg_{\alpha,\beta}, & x_{i,j} &\mapsto x_{i,j}, \\ \Pi_{\mathcal{F}} : S_{\mathcal{F}} := \mathbb{K}[T_{\alpha,\beta}] &\twoheadrightarrow \mathbb{K}[\mathbf{g}], & T_{\alpha,\beta} &\mapsto g_{\alpha,\beta}. \end{aligned}$$

Then $S_{\mathcal{R}}$ is a standard bigraded polynomial ring by setting $\deg x_{i,j} = (1, 0)$ and $\deg T_{\alpha,\beta} = (0, 1)$. We denote the defining ideal of $\mathcal{R}(I)$ by $\mathcal{J} = \ker \Pi_{\mathcal{R}}$ and that of $\mathbb{K}[\mathbf{g}]$ by $\mathcal{K} = \ker \Pi_{\mathcal{F}}$. We also consider the defining ideal \mathcal{L} of the symmetric algebra of I , that is, the sub-ideal of \mathcal{J} generated by the elements of bidegree $(*, 1)$.

Notation 2.2. We adopt the useful conventions $T_{\alpha,\beta} := -T_{\beta,\alpha}$ if $\alpha > \beta$, and $T_{\alpha,\alpha} := 0$.

Since the matrices \mathbf{X} and \mathbf{M} differ by a permutation, the isomorphism Θ between the two presentation rings of $\mathcal{R}(I)$ such that $\Psi_{\mathcal{R}} = \Pi_{\mathcal{R}} \circ \Theta$ is defined by a relabeling of the variables with some multiplications by -1 ; similarly for $\mathbb{K}[\mathbf{g}]$. To be more precise, let τ be the permutation of $\{1, \dots, c\}$ that we apply to the columns of \mathbf{X} to obtain \mathbf{M} . The

isomorphism Θ is given by the correspondence $Y_{\alpha,\beta} \mapsto T_{\tau(\alpha),\tau(\beta)}$ for all $\alpha < \beta$. Note that there is a -1 whenever $\tau(\alpha) > \tau(\beta)$.

Now we translate the polynomial relations (1), (2), (3) to this new presentation. It is easy to see that Θ preserves the form of the linear syzygies and the Plücker equations, i.e. the ideal \mathcal{L} is generated by the polynomials

$$(4) \quad L_{\alpha,\beta,\gamma} := \mu_{1,\alpha}T_{\beta,\gamma} - \mu_{1,\beta}T_{\alpha,\gamma} + \mu_{1,\gamma}T_{\alpha,\beta},$$

$$(5) \quad M_{\alpha,\beta,\gamma} := \mu_{2,\alpha}T_{\beta,\gamma} - \mu_{2,\beta}T_{\alpha,\gamma} + \mu_{2,\gamma}T_{\alpha,\beta},$$

for $(\alpha, \beta, \gamma) \in \binom{[c]}{3}$, while the Plücker relations are

$$(6) \quad P_{\alpha,\beta,\gamma,\delta} := T_{\alpha,\beta}T_{\gamma,\delta} - T_{\alpha,\gamma}T_{\beta,\delta} + T_{\alpha,\delta}T_{\beta,\gamma}$$

for $(\alpha, \beta, \gamma, \delta) \in \binom{[c]}{4}$. However, we need a notation for the relations (3).

Notation 2.3. For $\alpha \leq c - d$ define $\bar{\alpha} := \tau(\tau^{-1}(\alpha) + 1)$, that is, the next column in \mathbf{M} involving the same set of variables X_i as α .

The relations (3) correspond then to

$$(7) \quad Q_{\alpha,\beta,\gamma,\delta} := T_{\alpha,\beta}T_{\bar{\gamma},\bar{\delta}} - T_{\alpha,\gamma}T_{\bar{\beta},\bar{\delta}} + T_{\alpha,\delta}T_{\bar{\beta},\bar{\gamma}} + T_{\beta,\gamma}T_{\bar{\alpha},\bar{\delta}} - T_{\beta,\delta}T_{\bar{\alpha},\bar{\gamma}} + T_{\gamma,\delta}T_{\bar{\alpha},\bar{\beta}}$$

and the condition on the four indices becomes $(\alpha, \beta, \gamma, \delta) \in \binom{[c-d]}{4}$.

Since the isomorphism Θ is defined by a relabeling of the variables with some changes of sign, it allows to transfer not just relations but term orders as well. We will prove the Main Theorem using the presentations $\Pi_{\mathcal{R}}, \Pi_{\mathcal{F}}$ and the equations (4), (5), (6), (7).

3. TERM ORDER

In this section we define a term order on the presentation rings $S_{\mathcal{R}}$ and $S_{\mathcal{F}}$ of Section 2. It plays a key role in the proof of the Main Theorem. The construction is obtained by refining a lexicographic order with certain (multi)gradings, a generalization of the refinement of a term order by a non-negative weight ω [34].

Definition 3.1 (The term order \prec). We start by ordering the variables of $S_{\mathcal{R}}$ as follows:

- $T_{\alpha,\beta} \succ T_{\gamma,\delta}$ if $\alpha < \gamma$ or $\alpha = \gamma, \beta < \delta$;
- $x_{i,j} \succ x_{k,\ell}$ if $j < n_i, \ell < n_k$ and either $j < \ell$ or $j = \ell, i < k$;
- $x_{i,j} \succ x_{k,n_k}$ if $j < n_i$ or $j = n_i, i > k$;
- $x_{i,j} \succ T_{\alpha,\beta}$ for all i, j, α, β .

With this total order on the variables, we take the lexicographic term order on $S_{\mathcal{R}}$. We then refine the term order by a single grading defined by $\text{sdeg}(T_{\alpha,\beta}) = c - \alpha$. We further refine by a multigrading $\text{mdeg}(\cdot)$ defined as follows. Consider the semigroup $(\mathbb{N}^{c+d}, +)$ with canonical basis $\mathbf{e}_1 > \dots > \mathbf{e}_{c+d}$ and totally ordered by the lexicographic order. Set

$$\text{mdeg}(T_{\alpha,\beta}) = \mathbf{e}_{\alpha+d} + \mathbf{e}_{\beta+d}, \quad \text{mdeg}(\mu_{2,\gamma}) = \mathbf{e}_{\gamma+d}, \quad \text{mdeg}(x_{i,0}) = \mathbf{e}_i$$

The resulting term order is our desired \prec . On the subring $S_{\mathcal{F}} \subseteq S_{\mathcal{R}}$ we consider the restriction of this term order, which we also denote by \prec .

Remark 3.2. The total order on the variables $x_{i,j}$ is obtained by starting with $x_{1,0} \succ x_{2,0} \succ \cdots \succ x_{d,0}$ and then appending the remaining variables with the order they appear on the second row of \mathbf{M} . On the first row of \mathbf{M} we have

$$\mu_{1,1} \succ \mu_{1,2} \succ \cdots \succ \mu_{1,c-d} \succ \mu_{1,c-d+1} \prec \mu_{1,c-d+2} \prec \cdots \prec \mu_{1,c}.$$

We proceed with the calculation of the leading monomials of the polynomials in Section 2 with respect to \prec . Since \prec is the only term order we consider in this paper, we denote leading monomials just by $\text{LM}(\cdot)$.

Proposition 3.3. *We have*

$$\text{LM}(P_{\alpha,\beta,\gamma,\delta}) = T_{\alpha,\gamma}T_{\beta,\delta}, \quad \text{LM}(Q_{\alpha,\beta,\gamma,\delta}) = T_{\alpha,\beta}T_{\overline{\gamma},\overline{\delta}}, \quad \text{LM}(M_{\alpha,\beta,\gamma}) = \mu_{2,\beta}T_{\alpha,\gamma},$$

while $\text{LM}(L_{\alpha,\beta,\gamma})$ is the monomial containing the $x_{i,j}$ with lowest j and then lowest i .

Proof. For each polynomial, we must start by considering $\text{mdeg}(\cdot)$. The polynomials $P_{\alpha,\beta,\gamma,\delta}$ and $M_{\alpha,\beta,\gamma}$ are homogeneous with respect to $\text{mdeg}(\cdot)$, so the refinement has no effect on them. On the other hand, $Q_{\alpha,\beta,\gamma,\delta}$ and $L_{\alpha,\beta,\gamma}$ are not homogeneous. The unique monomial of highest multidegree in the support of $Q_{\alpha,\beta,\gamma,\delta}$ is $T_{\alpha,\beta}T_{\overline{\gamma},\overline{\delta}}$ and hence we can already conclude that it is the leading monomial.

For $L_{\alpha,\beta,\gamma}$ we distinguish two cases. If some $x_{i,0}$ appears in $L_{\alpha,\beta,\gamma}$ then the unique monomial of highest multidegree contains the $x_{i,0}$ with lowest i . If all the $x_{i,j}$ appearing in $L_{\alpha,\beta,\gamma}$ have $j > 0$ then $\alpha = \overline{\alpha_1}, \beta = \overline{\beta_1}, \gamma = \overline{\gamma_1}$ for some $\alpha_1, \beta_1, \gamma_1$; it follows that the unique monomial of highest multidegree contains $x_{i,j} = \mu_{2,\varepsilon}$ where $\varepsilon = \min\{\alpha_1, \beta_1, \gamma_1\}$, which is also the $x_{i,j}$ with lowest j and then lowest i . In either case, we conclude that $\text{LM}(L_{\alpha,\beta,\gamma})$ is the desired term.

The homogeneous component of highest degree with respect to $\text{sdeg}(\cdot)$ is $-T_{\alpha,\gamma}T_{\beta,\delta} + T_{\alpha,\delta}T_{\beta,\gamma}$ for $P_{\alpha,\beta,\gamma,\delta}$ and $-\mu_{2,\beta}T_{\alpha,\gamma} + \mu_{2,\gamma}T_{\alpha,\beta}$ for $M_{\alpha,\beta,\gamma}$. Finally, we break ties using the lexicographic order: we get $\text{LM}(P_{\alpha,\beta,\gamma,\delta}) = T_{\alpha,\gamma}T_{\beta,\delta}$ and $\text{LM}(M_{\alpha,\beta,\gamma}) = \mu_{2,\beta}T_{\alpha,\gamma}$. \square

Remark 3.4. From Remark 3.2 and Proposition 3.3 it follows that $\text{LM}(L_{\alpha,\beta,\gamma}) = \mu_{1,\alpha}T_{\beta,\gamma}$ or $\mu_{1,\gamma}T_{\alpha,\beta}$. Furthermore, we can only have $\text{LM}(L_{\alpha,\beta,\gamma}) = \mu_{1,\gamma}T_{\alpha,\beta}$ when $\gamma > c - d$.

4. THE INITIAL COMPLEX OF THE SPECIAL FIBER

In this section we study the flag simplicial complex Δ whose Stanley-Reisner ideal $\mathcal{I}_\Delta \subseteq S_{\mathcal{F}} = \mathbb{K}[T_{\alpha,\beta}]$ is generated by the monomials

$$\begin{aligned} (\dagger) \quad & \text{LM}(P_{\alpha,\beta,\gamma,\delta}) = T_{\alpha,\gamma}T_{\beta,\delta} & (\alpha, \beta, \gamma, \delta) \in \binom{[c]}{4}, \\ (\ddagger) \quad & \text{LM}(Q_{\alpha,\beta,\gamma,\delta}) = T_{\alpha,\beta}T_{\overline{\gamma},\overline{\delta}} & (\alpha, \beta, \gamma, \delta) \in \binom{[c-d]}{4}. \end{aligned}$$

At the end of this section we will prove that this is indeed an initial complex of the special fiber ring $\mathbb{K}[\mathbf{g}]$, namely $\mathcal{I}_\Delta = \text{in}_\prec(\mathcal{K})$. Our goal is to show that Δ is pure, compute its dimension and number of facets. We refer to [25] for background on simplicial complexes.

We interpret the elements of the vertex set $V = \{(\alpha, \beta) \mid 1 \leq \alpha < \beta \leq c\}$ of Δ as open intervals in the real line \mathbb{R} . The complex has different behavior in the two cases

$c < d + 4$ and $c \geq d + 4$, so we treat them separately. In the former case the minors of \mathbf{M} parametrize the Grassmann variety of lines in \mathbb{P}^{c-1} , cf. Remark 4.10.

4.1. Grassmann case: $c < d + 4$. We only have the generators (\dagger) . The faces of Δ are the subsets $F \subseteq V$ satisfying the “non-crossing” condition

$$(\diamond) \quad \text{for all } \mathfrak{I}_1, \mathfrak{I}_2 \in F \text{ either } \mathfrak{I}_1 \cap \mathfrak{I}_2 = \emptyset, \mathfrak{I}_1 \subseteq \mathfrak{I}_2, \text{ or } \mathfrak{I}_2 \subseteq \mathfrak{I}_1.$$

By maximality, a facet F of Δ is a face such that $(1, c) \in F$ and if $(\alpha, \beta) \in F$ with $\beta - \alpha > 1$ then there exists $\alpha < \gamma < \beta$ with $(\alpha, \gamma), (\gamma, \beta) \in F$.

Denote by $C_n = \binom{2n}{n} - \binom{2n}{n+1}$ the n -th Catalan number.

Proposition 4.1. *The complex Δ is pure of dimension $2c - 4$ and has C_{c-2} facets.*

Proof. Let F be a facet and $1 < \gamma < c$ with $(1, \gamma), (\gamma, c) \in F$. Then $F = F_1 \cup F_2 \cup \{(1, c)\}$ where $F_1 = \{\mathfrak{I} \in F \mid \mathfrak{I} \subseteq (1, \gamma)\}$ and $F_2 = \{\mathfrak{I} \in F \mid \mathfrak{I} \subseteq (\gamma, c)\}$. The subsets F_1 and F_2 correspond to facets of the smaller complexes obtained when \mathbf{M} has respectively γ and $c - \gamma$ columns. Both statements follow then by induction; for the number of facets, we use the well-known recursion $C_{n+1} = \sum_{i=0}^n C_i C_{n-i}$. \square

4.2. Non Grassmann case: $c \geq d + 4$. Now both sets of generators (\dagger) and (\ddagger) are nonempty. The faces of Δ still satisfy (\diamond) but are subject to more constraints. We say that an interval is *unitary* if it is of the form $(\alpha, \alpha + 1)$ for some α .

Lemma 4.2. *Let F be a facet of Δ . The minimal intervals of F with respect to inclusion are unitary.*

Proof. By (\diamond) the minimal intervals with respect to inclusion are disjoint. If (α, β) is a minimal interval of F , then it is easy to see that $\{(\alpha, \alpha + 1)\} \cup F$ or $\{(\beta - 1, \beta)\} \cup F$ is also a face, hence necessarily $\beta = \alpha + 1$. \square

Remark 4.3. Let $F \subseteq V$ be satisfying (\diamond) , so that F contains no minimal non-face of type (\dagger) . If F contains a minimal non-face of type (\ddagger) , then it contains one that consists of two minimal intervals of F . In other words, in order to characterize the facets of Δ , it suffices to characterize the possible unitary intervals contained in a facet. Observe that, unlike the Grassmann case, facets do not contain all the unitary intervals.

Lemma 4.4. *Let Σ_α be the set of facets whose leftmost minimal interval is $(\alpha, \alpha + 1)$.*

- If $\alpha \leq c - d - 2$ then there exists an integer $2 \leq \ell \leq d + 1$ such that the minimal intervals in every facet of Σ_α are precisely

$$\{(\beta, \beta + 1) \mid \beta \in \{\alpha, \dots, \alpha + \ell\} \cup \{c - d + \ell - 1, \dots, c - 1\}\}.$$

- If $\alpha > c - d - 2$ then $\Sigma_\alpha = \emptyset$.

Proof. Assume that $\alpha \leq c - d - 2$. A unitary interval $(\beta, \beta + 1)$ belongs to a facet $F \in \Sigma_\alpha$ if and only if there exists no column γ in \mathbf{M} with variables from the same set X_i as β such that $\alpha + 1 < \gamma < \beta$, and no column δ with variables from the same set X_j as $\beta + 1$ such that $\alpha + 1 < \delta < \beta + 1$. For each $i = 1, \dots, d$ take the first column $\gamma_i > \alpha + 1$ involving variables from X_i ; they are well defined because $\alpha \leq c - d - 2$. By construction

of \mathbf{M} , the γ_i 's split in two sequences of consecutive integers: $\{\alpha + 2, \dots, \alpha + \ell\}$ and $\{c - d + \ell, \dots, c\}$ for some $2 \leq \ell \leq d + 1$ (the latter sequence may be empty). It follows that $F \cup F' \cup F''$ is still a face of Δ , where $F' = \{(\alpha + 1, \alpha + 2), \dots, (\alpha + \ell, \alpha + \ell + 1)\}$, $F'' = \{(c - d + \ell - 1, c - d + \ell), \dots, (c - 1, c)\}$, hence necessarily $F', F'' \subseteq F$.

Finally, if the leftmost minimal interval of a face G is $(\alpha, \alpha + 1)$ for some $\alpha > c - d - 2$ then $G \cup \{(c - d - 2, c - d - 1)\}$ is also a face of Δ , so G is not a facet and $\Sigma_\alpha = \emptyset$. \square

So the facets of Δ can be characterized as the maximal subsets of V that satisfy (\diamond) and contain specified unitary intervals, as described in Lemma 4.4.

Proposition 4.5. *The complex Δ is pure of dimension $c + d - 1$.*

Proof. Let F be a facet of Δ . The Hasse diagram of the poset (F, \subseteq) is a rooted binary tree with root $(1, c)$. By Lemmas 4.2 and 4.4 this tree has $d + 2$ leaves, and hence $d + 1$ nodes with two children. The nodes with one child correspond to pairs $\mathfrak{J}_1, \mathfrak{J}_2 \in F$ such that $\mathfrak{J}_1 \subseteq \mathfrak{J}_2$, the length of $\mathfrak{J}_2 \setminus \mathfrak{J}_1$ is 1 and the unitary interval in $\mathfrak{J}_2 \setminus \mathfrak{J}_1$ is not in F : these pairs are in bijective correspondence with the unitary intervals outside F , therefore the number of such nodes is $(c - 1) - (d + 2)$. We conclude that the tree has $c + d$ nodes. \square

Now we want to compute the cardinality of Σ_α ; as we shall see, it only depends on α, c, d but not on ℓ . To this end we consider a more general object. Let $\alpha, \beta_1, \beta_2, \gamma$ be non-negative integers with $\beta_1 > 0$, $\alpha + \beta_1 + \beta_2 + \gamma = c - 1$, and such that $\beta_2 = 0$ if $\gamma = 0$. We split the unitary intervals of $(1, c)$ in four strings of two different colors: the first α unitary intervals are white, the next β_1 are black, the next γ are white, and the last β_2 are black.



Let $\Sigma(\alpha, \beta_1, \gamma, \beta_2)$ be the set of all $F \subseteq V$ that satisfy (\diamond) , whose unitary intervals are exactly the black intervals, and are maximal with respect to these properties. By Lemma 4.4 we have $\Sigma_\alpha = \Sigma(\alpha - 1, \ell + 1, c - \alpha - d - 2, d - \ell + 1)$ for some $2 \leq \ell \leq d + 1$.

For the enumeration of $\Sigma(\alpha, \beta_1, \gamma, \beta_2)$ we use a three-variate generalization of the Catalan numbers known as the *Catalan trapezoids* [28]. Define

$$C_m(n, k) = \begin{cases} \binom{n+k}{k} & \text{if } 0 \leq k < m, \\ \binom{n+k}{k} - \binom{n+k}{k-m} & \text{if } m \leq k < n + m - 1, \\ 0 & \text{if } k > n + m - 1. \end{cases}$$

Lemma 4.6. *We have $\text{Card}(\Sigma(\alpha, \beta_1, \gamma, \beta_2)) = C_{\alpha+1}(\beta_1 + \gamma + \beta_2 - 1, \alpha + \beta_1 + \beta_2 - 1)$ and in particular*

$$\text{Card}(\Sigma_\alpha) = \binom{c + d - 1}{\alpha + d} - \binom{c + d - 1}{d}.$$

Proof. The second statement is a direct consequence of the first and Lemma 4.4.

We claim that $\text{Card}(\Sigma(\alpha, \beta_1, \gamma, \beta_2)) = \chi(\alpha, \beta_1 + \beta_2, \gamma)$ for some function $\chi : \mathbb{N}^3 \rightarrow \mathbb{N}$ satisfying $\chi(\alpha, 1, 0) = \chi(0, 1, \gamma) = 1$ and if $\gamma > 0, \beta > 1$

$$(*) \quad \chi(\alpha, \beta, \gamma) = \chi(\alpha, \gamma - 1, \beta) + \chi(\alpha, \gamma + 1, \beta - 1).$$

From this the first statement follows, as the numbers $C_m(n, k)$ are characterized by the recursion $C_m(n, k) = C_m(n-1, k) + C_m(n, k-1)$ and the boundary conditions $C_m(n, 0) = 1$, $C_m(0, k) = 1$ for $k \leq m-1$, and $C_m(n, k) = 0$ for $k > n+m-1$, cf. [28].

For any $F \in \Sigma(\alpha, 1, \gamma, 0)$ the poset (F, \subseteq) is a saturated chain of intervals starting at $(\alpha+1, \alpha+2)$ and ending at $(1, c)$. For each pair $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$ of consecutive intervals in this chain, \mathfrak{I}_1 and \mathfrak{I}_2 share one endpoint. The left endpoint is shared by two consecutive intervals exactly γ times, and the right endpoint exactly α times. We get a bijection between $\Sigma(\alpha, 1, \gamma, 0)$ and the set of lattice paths from $(0, 0)$ to (α, γ) , therefore $\text{Card}(\Sigma(\alpha, 1, \gamma, 0)) = \binom{\alpha+\gamma}{\alpha}$. This verifies the claim when $\beta_1 = 1, \beta_2 = 0$.

Now suppose $\beta_1 > 1$. The case $\Sigma(0, \beta_1, 0, 0)$ is exactly the Grassmann case, where all the unitary intervals belong to each facet, therefore $\text{Card}(\Sigma(0, \beta_1, 0, 0)) = C_{\beta_1-1}$. Assume $\alpha > 0$, we compute $\text{Card}(\Sigma(\alpha, \beta_1, 0, 0))$ using a binary partition. Let \mathfrak{w} be the last white unitary interval and \mathfrak{b} the first black one:

$$\square \square \dots \square \overset{\mathfrak{w}}{\square} \overset{\mathfrak{b}}{\blacksquare} \blacksquare \dots \blacksquare \blacksquare.$$

For every $F \in \Sigma(\alpha, \beta_1, 0, 0)$ exactly one of the following two cases occurs:

- (1) $F \setminus \{\mathfrak{b}\} \in \Sigma(\alpha+1, \beta_1-1, 0, 0)$. Equivalently: every interval in F of length > 1 containing \mathfrak{b} also contains some other black unitary interval. Equivalently: the interval \mathfrak{I} of length 2 that contains $\mathfrak{b}, \mathfrak{w}$ is not in F .
- (2) $F \setminus \{\mathfrak{b}\} \notin \Sigma(\alpha+1, \beta_1-1, 0, 0)$. Equivalently: there exists an interval in F of length > 1 containing only the black interval \mathfrak{b} . Equivalently: the interval \mathfrak{I} of length 2 that contains $\mathfrak{b}, \mathfrak{w}$ is in F .

We define two separate bijections in the two cases.

- (1) We associate $F \mapsto F \setminus \{\mathfrak{b}\}$, i.e. “we make \mathfrak{b} white”. This gives a map from case (1) of $\Sigma(\alpha, \beta_1, 0, 0)$ to $\Sigma(\alpha+1, \beta_1-1, 0, 0)$. We define a map in the opposite direction by $G \mapsto G \cup \{\mathfrak{b}\}$. These two maps are inverse to each other, and thus we have a bijection between the two sets.
- (2) We associate $F \mapsto \tilde{F}$ obtained by throwing \mathfrak{I} away and collapsing \mathfrak{w} . Now this gives a map from case (2) of $\Sigma(\alpha, \beta_1, 0, 0)$ to $\Sigma(\alpha-1, \beta_1, 0, 0)$. We define a map in the opposite direction by adding a white unitary interval \mathfrak{w} next to \mathfrak{b} and adding the interval \mathfrak{I} . Again, these 2 maps are inverse to each other, and we have a bijection between the two sets.

We deduce that $\text{Card}(\Sigma(\alpha, \beta_1, 0, 0)) = \text{Card}(\Sigma(\alpha+1, \beta_1-1, 0, 0)) + \text{Card}(\Sigma(\alpha-1, \beta_1, 0, 0))$ and by induction this verifies the claim when $\gamma = \beta_2 = 0$.

Now suppose that $\beta_1 > 1$ and $\gamma > 0$; we compute $\text{Card}(\Sigma(\alpha, \beta_1, \gamma, \beta_2))$ by means of the same binary partition as in the previous step, but on a different pair of unitary intervals $\mathfrak{b}, \mathfrak{w}$. Namely, we distinguish the rightmost black unitary interval \mathfrak{b} which is adjacent to a white unitary interval \mathfrak{w} . If $\beta_2 = 0$ we have

$$\square \square \dots \square \blacksquare \blacksquare \dots \blacksquare \overset{\mathfrak{b}}{\blacksquare} \overset{\mathfrak{w}}{\square} \square \dots \square.$$

whereas if $\beta_2 > 0$ we have

$$\square\square\dots\square\square\square\dots\square\square\square\dots\square\square\square\dots\square.$$

For every $F \in \Sigma(\alpha, \beta_1, \gamma, \beta_2)$ we have two cases analogous to (1), (2) as above, and the same two bijections yield

$$\begin{aligned} \text{Card}(\Sigma(\alpha, \beta_1, \gamma, \beta_2)) &= \text{Card}(\Sigma(\alpha, \beta_1, \gamma - 1, \beta_2)) + \text{Card}(\Sigma(\alpha, \beta_1, \gamma + 1, \beta_2 - 1)) \text{ if } \beta_2 > 0, \\ \text{Card}(\Sigma(\alpha, \beta_1, \gamma, 0)) &= \text{Card}(\Sigma(\alpha, \beta_1, \gamma - 1, 0)) + \text{Card}(\Sigma(\alpha, \beta_1 - 1, \gamma + 1, 0)). \end{aligned}$$

By induction, the proof of the claim is completed. \square

Corollary 4.7. *The number of facets of Δ is*

$$\sum_{\alpha=1}^{c-d-2} \binom{c+d-1}{\alpha+d} - (c-d-2) \binom{c+d-1}{d}.$$

4.3. Gröbner basis for $\mathbb{K}[\mathbf{g}]$. We combine the results on Δ with those from [5, 11] to prove the main theorem of this section.

Theorem 4.8. *The defining ideal \mathcal{K} of the special fiber ring of I is minimally generated by the polynomials $P_{\alpha,\beta,\gamma,\delta}$ with $(\alpha, \beta, \gamma, \delta) \in \binom{[c]}{4}$ and $Q_{\alpha,\beta,\gamma,\delta}$ with $(\alpha, \beta, \gamma, \delta) \in \binom{[c-d]}{4}$, and they form a Gröbner basis with respect to the term order \prec .*

Proof. Since $P_{\alpha,\beta,\gamma,\delta}, Q_{\alpha,\beta,\gamma,\delta} \in \mathcal{K}$ the inclusion $\mathcal{I}_\Delta \subseteq \text{in}_\prec(\mathcal{K})$ holds. The ideal \mathcal{I}_Δ is unmixed as Δ is a pure simplicial complex by Propositions 4.1 and 4.5; it follows from the associativity formula for multiplicities that $\mathcal{I}_\Delta = \text{in}_\prec(\mathcal{K})$ if and only if the factor rings $S_{\mathcal{F}}/\mathcal{I}_\Delta$ and $S_{\mathcal{F}}/\text{in}_\prec(\mathcal{K})$ have the same Krull dimension and multiplicity. In other words, it suffices to show that $\dim(S_{\mathcal{F}}/\mathcal{I}_\Delta) = \dim(\mathbb{K}[\mathbf{g}])$ and $e(S_{\mathcal{F}}/\mathcal{I}_\Delta) = e(\mathbb{K}[\mathbf{g}])$.

These invariants are determined for the Stanley-Reisner ring $S_{\mathcal{F}}/\mathcal{I}_\Delta$ in Propositions 4.1, 4.5 and Corollary 4.7, and they only depend on c, d . They agree with those of $\mathbb{K}[\mathbf{g}]$ when the scroll $\mathcal{S}_{n_1, \dots, n_d}$ is balanced [11, Section 4]. However, by [5, Theorem 3.7] the Hilbert function and hence the multiplicity of the special fiber ring of a scroll depend only on c, d , and thus we get the equality $\mathcal{I}_\Delta = \text{in}_\prec(\mathcal{K})$ for any rational normal scroll.

We conclude that $\{P_{\mathbf{a}}, Q_{\mathbf{b}} \mid \mathbf{a} \in \binom{[c]}{4}, \mathbf{b} \in \binom{[c-d]}{4}\}$ is a Gröbner basis of \mathcal{K} , and in particular a (minimal) set of generators. \square

Recall that a standard graded \mathbb{K} -algebra A is *Koszul* if $\text{Tor}_i^A(\mathbb{K}, \mathbb{K})_j = 0$ whenever $i \neq j$, and that this condition holds whenever A is presented by a Gröbner basis of quadrics. We refer to [10] for details.

Corollary 4.9. *The special fiber ring of I is a Koszul algebra.*

Remark 4.10. In the special case when the ideal \mathcal{K} is only generated by the Plücker relations, i.e. when $c < d + 4$, $\mathbb{K}[\mathbf{g}]$ is the coordinate ring of the Grassmann variety $\mathbb{G}(1, c-1) \subseteq \mathbb{P}^{\binom{c}{2}-1}$. Our initial complex Δ is different from the classical one in the theory of straightening laws [25, 35]: in that context the leading monomial of $P_{\alpha,\beta,\gamma,\delta}$ is $T_{\alpha,\delta}T_{\beta,\gamma}$ and the resulting initial complex is the order complex of a poset; cf. [27, 32] for related considerations.

The initial complex Δ' constructed in [11] for balanced scrolls also follows the classical choice, and in fact it is different from our Δ for all values $c \geq 4, d \geq 1$. For example, it can be seen that Δ' has 4 cone points if $c < d + 4$, 2 if $c = d + 4$, and none if $c > d + 4$, whereas Δ has c cone points if $c < d + 4$, $c - 2$ if $c = d + 4$, and at least one if $c \geq d + 4$.

5. THE DEFINING EQUATIONS OF THE REES RING

It is proved in [5, Theorem 3.7] that the ideal I is of *fiber type*, that is, \mathcal{J} is generated in bidegrees $(*, 1)$ and $(0, *)$. In other words, $\mathcal{R}(I)$ is defined by the equations of $\mathbb{K}[\mathbf{g}]$ and of the symmetric algebra of I . Combining this with Theorem 4.8 we immediately obtain

Theorem 5.1. *The defining ideal \mathcal{J} of the Rees ring of I is minimally generated by the polynomials $P_{\alpha,\beta,\gamma,\delta}$ with $(\alpha, \beta, \gamma, \delta) \in \binom{[c]}{4}$, $Q_{\alpha,\beta,\gamma,\delta}$ with $(\alpha, \beta, \gamma, \delta) \in \binom{[c-d]}{4}$, and $L_{\alpha,\beta,\gamma}, M_{\alpha,\beta,\gamma}$ with $(\alpha, \beta, \gamma) \in \binom{[c]}{3}$.*

The goal of this section is to show that these polynomials form a Gröbner basis for \mathcal{J} with respect to \prec . Since, unlike the case of \mathcal{K} , we already know they generate the ideal, we can accomplish it by means of the Buchberger criterion. We refer to [25, 34] for background on Gröbner bases.

We need to show that all S-pairs reduce to 0. In general, S-pairs of polynomials with coprime leading monomial reduce to 0; hence, since we deal with quadrics, we consider linear S-pairs, i.e. pairs where the two leading monomials have exactly one variable in common. A priori there are $\binom{4+1}{2} = 10$ sets of S-pairs to examine, however, we will need to produce explicit equations of reduction only in 3 cases. The fact that S-pairs of type $S(P, P), S(P, Q), S(Q, Q)$ reduce to 0 was established in Theorem 4.8.

In subsection 5.1 we prove that the S-pairs of type $S(P, L), S(P, M), S(L, L), S(M, M)$, reduce to 0. A simple argument allows to reduce to 2×4 and 2×5 matrices: for each two polynomials in an S-pair, the relations involving the same set of columns form a (smaller) Gröbner basis. Up to renaming the variables, there are 26 such sets, and we verify the statement by a Hilbert series computation with Macaulay2.

In subsection 5.2 we describe six classes of linear syzygies of the polynomials in Theorem 5.1, five of which arise by expanding certain determinants. From these syzygies we derive equations of reduction for the S-pairs of type $S(Q, M), S(Q, L), S(L, M)$, thus concluding the proof.

5.1. Computer-assisted reduction of S-pairs. In this subsection we assume $\mathbb{Q} \subseteq \mathbb{K}$. It is not restrictive for our purpose, as we shall still be able to deduce Theorem 5.15 in arbitrary characteristic.

Lemma 5.2. *Let $\Gamma \subseteq [c]$ with $\text{Card}(\Gamma) \in \{4, 5\}$. The set $\{P_{\mathbf{a}}, M_{\mathbf{b}} \mid \mathbf{a} \in \binom{\Gamma}{4}, \mathbf{b} \in \binom{\Gamma}{3}\}$ is a Gröbner basis.*

Proof. Up to renaming the variables, we may assume that $\Gamma = [4]$ or $[5]$, notice that this preserves the leading monomials by Proposition 3.3. In order to prove the statement, we verify for $c = 4, 5$ that the Hilbert function of these polynomials agrees with the one of the leading monomials. This is checked on Macaulay2. \square

By Remark 3.4 $\text{LM}(L_{\alpha,\beta,\gamma})$ does not only depend on α, β, γ but also on the partition n_1, \dots, n_d , so we need to check more than one configuration of leading monomials in the next lemma.

Lemma 5.3. *Let $\Gamma \subseteq [c]$ with $\text{Card}(\Gamma) \in \{4, 5\}$. The set $\{P_{\mathbf{a}}, L_{\mathbf{b}} \mid \mathbf{a} \in \binom{[\Gamma]}{4}, \mathbf{b} \in \binom{[\Gamma]}{3}\}$ is a Gröbner basis.*

Proof. Suppose $\text{Card}(\Gamma) = 5$. Renaming the variables, the set consists of $T_{\alpha,\beta}T_{\gamma,\delta} - T_{\alpha,\gamma}T_{\beta,\delta} + T_{\alpha,\delta}T_{\beta,\gamma}$ with $(\alpha, \beta, \gamma, \delta) \in \binom{[5]}{4}$ and $x_{\alpha}T_{\beta,\gamma} - x_{\beta}T_{\alpha,\gamma} + x_{\gamma}T_{\alpha,\beta}$ with $(\alpha, \beta, \gamma) \in \binom{[5]}{3}$. The leading monomial of the former is $T_{\alpha,\gamma}T_{\beta,\delta}$, while for the latter it is the monomial with largest x variable with respect to \prec . By Remark 3.2 the order on x_1, \dots, x_5 is such that for any $(\alpha, \beta, \gamma) \in \binom{[5]}{3}$ if $x_{\beta} \succ x_{\gamma}$ then $x_{\alpha} \succ x_{\beta}$; there are 16 such orders. For each of them we verify on Macaulay2 that the two ideals of $\mathbb{K}[x_1, \dots, x_5, T_{1,2}, \dots, T_{4,5}]$ generated respectively by the polynomials and by the leading monomials have the same Hilbert function. The case $\text{Card}(\Gamma) = 4$ is analogous, with 8 orders to check. \square

Corollary 5.4. *The sets $\{P_{\mathbf{a}}, M_{\mathbf{b}} \mid \mathbf{a} \in \binom{[c]}{4}, \mathbf{b} \in \binom{[c]}{3}\}$ and $\{P_{\mathbf{a}}, L_{\mathbf{b}} \mid \mathbf{a} \in \binom{[c]}{4}, \mathbf{b} \in \binom{[c]}{3}\}$ are Gröbner bases.*

Proof. For any linear S-pair $S(P_{\alpha,\beta,\gamma,\delta}, M_{\varepsilon,\zeta,\eta})$ or $S(P_{\alpha,\beta,\gamma,\delta}, L_{\varepsilon,\zeta,\eta})$ we necessarily have $\text{Card}\{\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta\} \in \{4, 5\}$. By Lemmas 5.2, 5.3 the S-pairs reduce to 0. \square

The argument does not extend to the reduction of the remaining three S-pairs, where the $Q_{\alpha,\beta,\gamma,\delta}$ are needed. In fact, there seems to be no “small” Gröbner basis containing the equations $Q_{\alpha,\beta,\gamma,\delta}$. Nevertheless, the equations of reduction do not propagate along the matrix \mathbf{M} , but they only involve at most 10 columns each.

5.2. Linear syzygies of \mathcal{J} and S-pairs. We use a direct approach to show that the remaining S-pairs reduce to 0, based on the analysis of the linear syzygies of \mathcal{J} .

Denote by $\text{Supp}(\cdot)$ the set of monomials appearing in a polynomial, by \mathfrak{S}_n the symmetric group on $[n]$ and by $\sigma(\cdot)$ the sign of a permutation.

Remark 5.5. Suppose we have a polynomial ring with a term order $<$ and an equation

$$m_1 F_1 + \sum_{i=2}^M m_i F_i = n_1 G_1 + \sum_{j=2}^N n_j G_j$$

where m_i, n_j are monomials with signs, F_i, G_j are polynomials, and $S(F_1, G_1) = m_1 F - n_1 G_1$. Suppose the following conditions hold:

- $\text{Supp}(m_i F_i) \cap \text{Supp}(m_{i'} F_{i'}) = \text{Supp}(n_j G_j) \cap \text{Supp}(n_{j'} G_{j'}) = \emptyset$ for all $i \neq i', j = j'$;
- the highest two monomials in the left side appear in $\text{Supp}(m_1 F_1)$.

Then $S(F_1, G_1)$ reduces to 0 modulo $\{F_i, G_j\}$, because $S(F_1, G_1) = \sum_{j=2}^N n_j G_j - \sum_{i=2}^M m_i F_i$ with $\text{LM}(S(F_1, G_1)) \geq \text{LM}(n_j G_j), \text{LM}(m_i F_i)$ for all $i, j \geq 2$.

We adopt a more flexible notation, allowing arbitrary tuples as subscripts of the equations.

Notation 5.6. We extend Notation 2.2 to equations (4), (5), (6) by setting

$L_{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}} := \sigma(\mathbf{i})L_{\alpha_1, \alpha_2, \alpha_3}$, $M_{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}} := \sigma(\mathbf{i})M_{\alpha_1, \alpha_2, \alpha_3}$, $P_{\beta_{j_1}, \beta_{j_2}, \beta_{j_3}, \beta_{j_4}} := \sigma(\mathbf{j})P_{\beta_1, \beta_2, \beta_3, \beta_4}$
for every $(\alpha_1, \alpha_2, \alpha_3) \in \binom{[c]}{3}$, $(\beta_1, \beta_2, \beta_3, \beta_4) \in \binom{[d]}{4}$, $(i_1, i_2, i_3) \in \mathfrak{S}_3$, $(j_1, j_2, j_3, j_4) \in \mathfrak{S}_4$, and

$$L_{\alpha_1, \alpha_2, \alpha_3} := 0, \quad M_{\alpha_1, \alpha_2, \alpha_3} := 0, \quad P_{\beta_1, \beta_2, \beta_3, \beta_4} := 0$$

whenever $\text{Card}\{\alpha_1, \alpha_2, \alpha_3\} < 3$, $\text{Card}\{\beta_1, \beta_2, \beta_3, \beta_4\} < 4$.

Lemma 5.7. Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \binom{[c-d]}{5}$. We have two syzygies

$$(S1) \quad \sum \sigma(\mathbf{i})T_{\overline{\alpha_{i_1}}, \overline{\alpha_{i_2}}}M_{\alpha_{i_3}, \alpha_{i_4}, \alpha_{i_5}} = \sum \sigma(\mathbf{j})\mu_{2, \alpha_{j_1}}Q_{\alpha_{j_2}, \alpha_{j_3}, \alpha_{j_4}, \alpha_{j_5}}.$$

$$(S2) \quad \sum \sigma(\mathbf{i})T_{\alpha_{i_1}, \alpha_{i_2}}M_{\overline{\alpha_{i_3}}, \overline{\alpha_{i_4}}, \overline{\alpha_{i_5}}} = \sum \sigma(\mathbf{j})\mu_{2, \overline{\alpha_{j_1}}}Q_{\alpha_{j_2}, \alpha_{j_3}, \alpha_{j_4}, \alpha_{j_5}}.$$

where the sums range over all $\mathbf{i} = (i_1, i_2, i_3, i_4, i_5), \mathbf{j} = (j_1, j_2, j_3, j_4, j_5) \in \mathfrak{S}_5$ with $i_1 < i_2, i_3 < i_4 < i_5$ and $j_2 < j_3 < j_4 < j_5$. The highest three monomials in the left side of (S1) and (S2) are respectively those in $\text{Supp}(T_{\overline{\alpha_4}, \overline{\alpha_5}}M_{\alpha_1, \alpha_2, \alpha_3})$ and $\text{Supp}(T_{\alpha_1, \alpha_2}M_{\overline{\alpha_3}, \overline{\alpha_4}, \overline{\alpha_5}})$.

Proof. Both sides of (S1) are equal to $\sum \sigma(\mathbf{h})\mu_{2, \alpha_{h_1}}T_{\alpha_{h_2}, \alpha_{h_3}}T_{\overline{\alpha_{h_4}}, \overline{\alpha_{h_5}}}$ where the sum ranges over all $\mathbf{h} = (h_1, h_2, h_3, h_4, h_5) \in \mathfrak{S}_5$ with $h_2 < h_3, h_4 < h_5$. By Definition 3.1, the highest multidegree realized by these monomials is $\mathbf{e}_{\alpha_1+d} + \mathbf{e}_{\alpha_2+d} + \mathbf{e}_{\alpha_3+d} + \mathbf{e}_{\overline{\alpha_4}+d} + \mathbf{e}_{\overline{\alpha_5}+d}$. On the left side, this multidegree is realized by the monomials in the support of $T_{\overline{\alpha_4}, \overline{\alpha_5}}M_{\alpha_1, \alpha_2, \alpha_3}$.

Similarly, both sides of (S2) are equal to $\sum \sigma(\mathbf{h})\mu_{2, \overline{\alpha_{h_1}}}T_{\alpha_{h_2}, \alpha_{h_3}}T_{\overline{\alpha_{h_4}}, \overline{\alpha_{h_5}}}$. The highest multidegree is $\mathbf{e}_{\alpha_1+d} + \mathbf{e}_{\alpha_2+d} + \mathbf{e}_{\overline{\alpha_3}+d} + \mathbf{e}_{\overline{\alpha_4}+d} + \mathbf{e}_{\overline{\alpha_5}+d}$, and on the left side it is realized by the monomials in the support of $T_{\alpha_1, \alpha_2}M_{\overline{\alpha_3}, \overline{\alpha_4}, \overline{\alpha_5}}$.

Finally, we remark that in each side of (S1) and (S2) the supports are disjoint. \square

The syzygies (S1), (S2) arise from different expansions of the determinants

$$\begin{vmatrix} \mu_{2, \alpha_1} & \mu_{2, \alpha_2} & \mu_{2, \alpha_3} & \mu_{2, \alpha_4} & \mu_{2, \alpha_5} \\ u_{1, \alpha_1} & u_{1, \alpha_2} & u_{1, \alpha_3} & u_{1, \alpha_4} & u_{1, \alpha_5} \\ u_{2, \alpha_1} & u_{2, \alpha_2} & u_{2, \alpha_3} & u_{2, \alpha_4} & u_{2, \alpha_5} \\ w_{1, \overline{\alpha_1}} & w_{1, \overline{\alpha_2}} & w_{1, \overline{\alpha_3}} & w_{1, \overline{\alpha_4}} & w_{1, \overline{\alpha_5}} \\ w_{2, \overline{\alpha_1}} & w_{2, \overline{\alpha_2}} & w_{2, \overline{\alpha_3}} & w_{2, \overline{\alpha_4}} & w_{2, \overline{\alpha_5}} \end{vmatrix}, \quad \begin{vmatrix} \mu_{2, \overline{\alpha_1}} & \mu_{2, \overline{\alpha_2}} & \mu_{2, \overline{\alpha_3}} & \mu_{2, \overline{\alpha_4}} & \mu_{2, \overline{\alpha_5}} \\ u_{1, \alpha_1} & u_{1, \alpha_2} & u_{1, \alpha_3} & u_{1, \alpha_4} & u_{1, \alpha_5} \\ u_{2, \alpha_1} & u_{2, \alpha_2} & u_{2, \alpha_3} & u_{2, \alpha_4} & u_{2, \alpha_5} \\ w_{1, \overline{\alpha_1}} & w_{1, \overline{\alpha_2}} & w_{1, \overline{\alpha_3}} & w_{1, \overline{\alpha_4}} & w_{1, \overline{\alpha_5}} \\ w_{2, \overline{\alpha_1}} & w_{2, \overline{\alpha_2}} & w_{2, \overline{\alpha_3}} & w_{2, \overline{\alpha_4}} & w_{2, \overline{\alpha_5}} \end{vmatrix},$$

where the entries u 's, w 's are “copies” of the variables μ 's.

Lemma 5.8. Let $\alpha, \beta, \gamma \in [c-d]$. If $\overline{\alpha} < \overline{\beta} < \overline{\gamma}$ then $\alpha < \beta$ or $\gamma < \beta$.

Proof. Suppose that $\beta < \alpha, \gamma$. By construction of \mathbf{M} , $\overline{\beta} > \overline{\alpha}$ forces $\overline{\beta} > c-d$, but we cannot have $\beta < \gamma$ and $c-d < \overline{\beta} < \overline{\gamma}$, contradiction. \square

Proposition 5.9. The S -pairs of type $S(Q, M)$ reduce to 0.

Proof. Given a linear S -pair $S(Q_{\alpha, \beta, \gamma, \delta}, M_{\varepsilon, \zeta, \eta})$ with $(\alpha, \beta, \gamma, \delta) \in \binom{[c-d]}{4}$, $(\varepsilon, \zeta, \eta) \in \binom{[c]}{3}$, by Proposition 3.3 there are two cases. If $\{\varepsilon, \eta\} = \{\alpha, \beta\}$ then the conclusion follows from Remark 5.5 using (S1) with $(\alpha, \zeta, \beta, \gamma, \delta)$. If $\{\varepsilon, \eta\} = \{\overline{\gamma}, \overline{\delta}\}$ then it follows that $\zeta = \overline{\theta}$ for some θ . By Lemma 5.8 we have $\theta > \gamma$, and clearly $\theta \neq \delta$. The conclusion follows from Remark 5.5 using (S2) with $(\alpha, \beta, \gamma, \delta, \theta)$ or $(\alpha, \beta, \gamma, \theta, \delta)$. \square

Corollary 5.10. *The set $\{P_{\mathbf{a}}, Q_{\mathbf{b}}, M_{\mathbf{c}} \mid \mathbf{a} \in \binom{[c]}{4}, \mathbf{b} \in \binom{[c-d]}{4}, \mathbf{c} \in \binom{[c]}{3}\}$ is a Gröbner basis.*

Proof. It follows from Theorem 4.8, Corollary 5.4, Lemma 5.7, and Proposition 5.9. \square

By contrast, the sets $\{P_{\mathbf{a}}, Q_{\mathbf{b}}, L_{\mathbf{c}}\}, \{P_{\mathbf{a}}, L_{\mathbf{b}}, M_{\mathbf{c}}\}$ are not, in general, Gröbner bases.

Lemma 5.11. *Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \binom{[c-d]}{4}$ and $\varepsilon \in [c]$. We have three syzygies*

$$(S3) \quad \sum \sigma(\mathbf{i}) T_{\alpha_{i_1}, \alpha_{i_2}} L_{\overline{\alpha_{i_3}, \alpha_{i_4}}, \varepsilon} = Q_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} \mu_{1, \varepsilon} + \sum \sigma(\mathbf{j}) M_{\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}} T_{\overline{\alpha_{j_4}}, \varepsilon}$$

$$(S4) \quad \sum \sigma(\mathbf{i}) T_{\overline{\alpha_{i_1}, \alpha_{i_2}}} M_{\alpha_{i_3}, \alpha_{i_4}, \varepsilon} = Q_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} \mu_{2, \varepsilon} + \sum \sigma(\mathbf{j}) L_{\overline{\alpha_{j_1}, \alpha_{j_2}}, \overline{\alpha_{j_3}}} T_{\alpha_{j_4}, \varepsilon}$$

$$(S5) \quad \sum \sigma(\mathbf{i}) T_{\overline{\alpha_{i_1}, \alpha_{i_2}}} L_{\alpha_{i_3}, \alpha_{i_4}, \varepsilon} + \sum \sigma(\mathbf{h}) M_{\alpha_{h_1}, \alpha_{h_2}, \varepsilon} T_{\alpha_{h_3}, \overline{\alpha_{h_4}}} - \sum \sigma(\mathbf{j}) P_{\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}, \varepsilon} \mu_{2, \overline{\alpha_{j_4}}} + \\ \sum \sigma(\mathbf{j}) M_{\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}} T_{\overline{\alpha_{j_4}}, \varepsilon} = \mu_{1, \varepsilon} Q_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} + \\ \sum \sigma(\mathbf{h}) L_{\overline{\alpha_{h_1}, \alpha_{h_2}}, \overline{\alpha_{h_3}}} T_{\alpha_{h_4}, \varepsilon} + \sum \sigma(\mathbf{h}) T_{\alpha_{h_1}, \alpha_{h_2}} M_{\alpha_{h_3}, \overline{\alpha_{h_4}}, \varepsilon}$$

where the sums range over $\mathbf{i} = (i_1, i_2, i_3, i_4), \mathbf{j} = (j_1, j_2, j_3, j_4), \mathbf{h} = (h_1, h_2, h_3, h_4) \in \mathfrak{S}_4$ with $i_1 < i_2, i_3 < i_4, j_1 < j_2 < j_3, h_1 < h_2$.

Proof. Since $\mu_{1, \overline{\alpha_i}} = \mu_{2, \alpha_i}$, both sides of (S3) are equal to

$$\sum \sigma(i_1, i_2, i_3, i_4) T_{\alpha_{i_1}, \alpha_{i_2}} T_{\overline{\alpha_{i_3}, \alpha_{i_4}}} \mu_{1, \varepsilon} - \sum \sigma(j_1, j_2, j_3, j_4) T_{\alpha_{j_1}, \alpha_{j_2}} T_{\overline{\alpha_{j_3}, \varepsilon}} \mu_{1, \overline{\alpha_{j_4}}}$$

with sums ranging over all permutations with $i_1 < i_2, i_3 < i_4, j_1 < j_2$. Similarly, both sides of (S4) are equal to

$$\sum \sigma(i_1, i_2, i_3, i_4) T_{\overline{\alpha_{i_1}, \alpha_{i_2}}} T_{\alpha_{i_3}, \alpha_{i_4}} \mu_{2, \varepsilon} - \sum \sigma(j_1, j_2, j_3, j_4) T_{\overline{\alpha_{j_1}, \alpha_{j_2}}} T_{\alpha_{j_3}, \varepsilon} \mu_{2, \alpha_{j_4}}.$$

Both sides of (S5) are equal to

$$\sum \sigma(\mathbf{i}) T_{\overline{\alpha_{i_1}, \alpha_{i_2}}} T_{\alpha_{i_3}, \alpha_{i_4}} \mu_{1, \varepsilon} - \sum \sigma(\mathbf{j}) T_{\overline{\alpha_{j_1}, \alpha_{j_2}}} T_{\alpha_{j_3}, \varepsilon} \mu_{1, \alpha_{j_4}} + \sum \sigma(\mathbf{j}) T_{\alpha_{j_1}, \alpha_{j_2}} T_{\alpha_{j_3}, \overline{\alpha_{j_4}}} \mu_{2, \varepsilon} - \\ \sum \sigma(\mathbf{h}) T_{\alpha_{h_1}, \overline{\alpha_{h_2}}} T_{\alpha_{h_3}, \varepsilon} \mu_{2, \alpha_{h_4}} - \sum \sigma(\mathbf{j}) T_{\alpha_{j_1}, \alpha_{j_2}} T_{\alpha_{j_3}, \varepsilon} \mu_{2, \overline{\alpha_{j_4}}} - \sum \sigma(\mathbf{j}) T_{\alpha_{j_1}, \alpha_{j_2}} T_{\overline{\alpha_{j_3}, \varepsilon}} \mu_{2, \alpha_{j_4}}$$

with sums ranging over all permutations with $i_1 < i_2, i_3 < i_4, j_1 < j_2$.

We remark that the supports in each side of (S3), (S4), (S5) are disjoint. \square

The syzygies (S3), (S4) arise from appropriate expansions of the determinants

$$\begin{vmatrix} \mu_{2, \alpha_1} & \mu_{2, \alpha_2} & \mu_{2, \alpha_3} & \mu_{2, \alpha_4} & \mu_{1, \varepsilon} \\ u_{1, \alpha_1} & u_{1, \alpha_2} & u_{1, \alpha_3} & u_{1, \alpha_4} & 0 \\ u_{2, \alpha_1} & u_{2, \alpha_2} & u_{2, \alpha_3} & u_{2, \alpha_4} & 0 \\ w_{1, \overline{\alpha_1}} & w_{1, \overline{\alpha_2}} & w_{1, \overline{\alpha_3}} & w_{1, \overline{\alpha_4}} & u_{1, \varepsilon} \\ w_{2, \overline{\alpha_1}} & w_{2, \overline{\alpha_2}} & w_{2, \overline{\alpha_3}} & w_{2, \overline{\alpha_4}} & u_{2, \varepsilon} \end{vmatrix}, \quad \begin{vmatrix} \mu_{2, \alpha_1} & \mu_{2, \alpha_2} & \mu_{2, \alpha_3} & \mu_{2, \alpha_4} & \mu_{2, \varepsilon} \\ u_{1, \alpha_1} & u_{1, \alpha_2} & u_{1, \alpha_3} & u_{1, \alpha_4} & u_{1, \varepsilon} \\ u_{2, \alpha_1} & u_{2, \alpha_2} & u_{2, \alpha_3} & u_{2, \alpha_4} & u_{2, \varepsilon} \\ w_{1, \overline{\alpha_1}} & w_{1, \overline{\alpha_2}} & w_{1, \overline{\alpha_3}} & w_{1, \overline{\alpha_4}} & 0 \\ w_{2, \overline{\alpha_1}} & w_{2, \overline{\alpha_2}} & w_{2, \overline{\alpha_3}} & w_{2, \overline{\alpha_4}} & 0 \end{vmatrix}.$$

Lemma 5.12. *Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \binom{[c]}{4}$. We have the syzygy*

$$(S6) \quad \sum \sigma(\mathbf{i}) L_{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}} \mu_{2, \alpha_{i_4}} = - \sum \sigma(\mathbf{i}) M_{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}} \mu_{1, \alpha_{i_4}}$$

where the sums range over $\mathbf{i} = (i_1, i_2, i_3, i_4) \in \mathfrak{S}_4$ with $i_1 < i_2 < i_3$.

Proof. Both sides are equal to $\sum \sigma(\mathbf{i}) \mu_{1,\alpha_{i_1}} \mu_{2,\alpha_{i_2}} T_{\alpha_{i_3},\alpha_{i_4}}$ with sum over \mathbf{i} with $i_3 < i_4$. \square

The syzygy (S6) arises from the determinant

$$\begin{vmatrix} \mu_{1,\alpha_1} & \mu_{1,\alpha_2} & \mu_{1,\alpha_3} & \mu_{1,\alpha_4} \\ u_{1,\alpha_1} & u_{1,\alpha_2} & u_{1,\alpha_3} & u_{1,\alpha_4} \\ u_{2,\alpha_1} & u_{2,\alpha_2} & u_{2,\alpha_3} & u_{2,\alpha_4} \\ \mu_{2,\alpha_1} & \mu_{2,\alpha_2} & \mu_{2,\alpha_3} & \mu_{2,\alpha_4} \end{vmatrix}.$$

Proposition 5.13. *The S-pairs of type $S(Q, L)$ reduce to 0.*

Proof. Let $S(Q_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}, L_{\beta,\gamma,\delta})$ be a linear S-pair with $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \binom{[c-d]}{4}$, $(\beta, \gamma, \delta) \in \binom{[c]}{3}$, and ε be such that $\mu_{1,\varepsilon}$ divides $\text{LM}(L_{\beta,\gamma,\delta})$. There are two possibilities.

If $\text{LM}(L_{\beta,\gamma,\delta}) = \mu_{1,\varepsilon} T_{\alpha_1,\alpha_2}$ then $L_{\beta,\gamma,\delta} = L_{\alpha_1,\alpha_2,\varepsilon}$ and we use equation (S5). Since $\mu_{1,\varepsilon} \succ \mu_{1,\alpha_1}, \mu_{1,\alpha_2}$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \binom{[c-d]}{4}$ it follows that $\mu_{1,\varepsilon}$ is the highest variable among the $\mu_{i,j}$'s appearing in (S5). Hence the highest multidegree in (S5) is $\mathbf{e}_\varepsilon + \mathbf{e}_{\alpha_1+d} + \mathbf{e}_{\alpha_2+d} + \mathbf{e}_{\alpha_3+d} + \mathbf{e}_{\alpha_4+d}$, and on the left side of (S5) this multidegree is realized precisely by the monomials in $\text{Supp}(T_{\alpha_3,\alpha_4} L_{\alpha_1,\alpha_2,\varepsilon})$. By Remark 5.5 the S-pair reduces to 0.

If $\text{LM}(L_{\beta,\gamma,\delta}) = \mu_{1,\varepsilon} T_{\alpha_3,\alpha_4}$, then $L_{\beta,\gamma,\delta} = L_{\alpha_3,\alpha_4,\varepsilon}$ and we use equation (S3). The (not necessarily distinct) variables $\mu_{1,\varepsilon}, \mu_{2,\alpha_1}, \mu_{2,\alpha_2}$ are the highest among the $\mu_{i,j}$'s appearing in (S3), hence the highest multidegree in (S3) is again $\mathbf{e}_\varepsilon + \mathbf{e}_{\alpha_1+d} + \mathbf{e}_{\alpha_2+d} + \mathbf{e}_{\alpha_3+d} + \mathbf{e}_{\alpha_4+d}$. This multidegree is realized on the left side of (S3) by the monomials in $\text{Supp}(T_{\alpha_1,\alpha_2} L_{\alpha_3,\alpha_4,\varepsilon})$, and by Remark 5.5 the S-pair reduces to 0. \square

Proposition 5.14. *The S-pairs of type $S(L, M)$ reduce to 0.*

Proof. Given an S-pair $S(L_{\alpha,\beta,\gamma}, M_{\delta,\zeta,\varepsilon})$ with $(\alpha, \beta, \gamma), (\delta, \zeta, \varepsilon) \in \binom{[c]}{3}$, we distinguish two cases according to the bidegree.

Suppose the S-pair has bidegree $(1, 2)$, then $\gcd(\text{LM}(L_{\alpha,\beta,\gamma}), \text{LM}(M_{\delta,\zeta,\varepsilon})) = \mu_{2,\zeta}$, and by Remark 3.4 either $\bar{\zeta} = \alpha$ or $\bar{\zeta} = \gamma$. If for example $\bar{\zeta} = \alpha$ then $\mu_{2,\zeta} = \mu_{1,\alpha} \succ \mu_{1,\beta}, \mu_{1,\gamma}$ and by Proposition 3.3 there exist ι, θ such that $\beta = \bar{\theta}, \gamma = \bar{\iota}$; similarly if $\bar{\zeta} = \gamma$. In any case we get $\{\alpha, \beta, \gamma\} = \{\zeta, \iota, \theta\}$ for some $\iota \neq \theta$. We have $\mu_{2,\delta} \succ \mu_{2,\zeta} \succ \mu_{2,\iota}, \mu_{2,\theta}$ and therefore $\delta < \zeta < \theta, \iota$. Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \binom{[c-d]}{4}$ such that $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{\delta, \zeta, \theta, \iota\}$, so $M_{\delta,\zeta,\varepsilon} = \pm M_{\alpha_1,\alpha_2,\varepsilon}$, $L_{\alpha,\beta,\gamma} = \pm L_{\alpha_2,\alpha_3,\alpha_4}$ and we use equation (S4). The highest multidegree is $\mathbf{e}_{\varepsilon+d} + \mathbf{e}_{\alpha_1+d} + \mathbf{e}_{\alpha_2+d} + \mathbf{e}_{\alpha_3+d} + \mathbf{e}_{\alpha_4+d}$ and is realized on the left side by the monomials in $\text{Supp}(T_{\alpha_3,\alpha_4} M_{\alpha_1,\alpha_2,\varepsilon})$. By Remark 5.5 we deduce the desired conclusion.

Suppose now the S-pair has bidegree $(2, 1)$, then either $\{\delta, \varepsilon\} = \{\alpha, \beta\}$ or $\{\delta, \varepsilon\} = \{\beta, \gamma\}$. If $\{\delta, \varepsilon\} = \{\alpha, \beta\}$ then $\mu_{1,\gamma} \succ \mu_{1,\alpha}, \mu_{1,\beta}$ and also $\mu_{1,\gamma} \succ \mu_{1,\zeta}$ by Remark 3.2, since $\alpha < \zeta < \beta < \gamma$. We use equation (S6) with $(\alpha, \zeta, \beta, \gamma)$, and the three terms of highest multidegree in the left side are those in $\text{Supp}(\mu_{1,\zeta} L_{\alpha,\beta,\gamma})$. If $\{\delta, \varepsilon\} = \{\beta, \gamma\}$ then $\mu_{1,\alpha} \succ \mu_{1,\alpha}, \mu_{1,\beta}$ and we proceed in the same way using equation (S6) with $(\alpha, \beta, \zeta, \gamma)$. The supports in each side of (S6) are disjoint, so we conclude by Remark 5.5. \square

We combine the results obtained so far to prove the main theorem of this section.

Theorem 5.15. *The minimal generators of the defining ideal \mathcal{J} of the Rees ring of I from a Gröbner basis with respect to \prec .*

Proof. Assume first that $\text{char}(\mathbb{K}) = 0$. Then the conclusion follows from Buchberger's criterion, since all the S-pairs among generators of \mathcal{J} reduce to 0. Equivalently, the Hilbert function of the ideal generated by the leading monomials of the generators equals the one of \mathcal{J} . However, the Hilbert function of a monomial ideal does not depend on $\text{char}(\mathbb{K})$, while it was shown in [5, Theorem 3.7] that the Hilbert function of \mathcal{J} does not depend on $\text{char}(\mathbb{K})$ either. Thus the statement holds in arbitrary characteristic. \square

Corollary 5.16. *The Rees ring of I is a Koszul algebra.*

Remark 5.17. Let $\mathcal{S}_{n_1, \dots, n_d} \subseteq \mathbb{P}^{c+d-1}$ be a rational normal scroll and $I = (\mathbf{g})$ its homogeneous ideal. The Main Theorem implies that, expressing $\mathcal{R}(I)$ and $\mathbb{K}[\mathbf{g}]$ as quotients of $S_{\mathcal{R}} = \mathbb{K}[x_1, \dots, x_{c+d}, T_1, \dots, T_{\binom{c-1}{2}}]$ and $S_{\mathcal{F}} = \mathbb{K}[T_1, \dots, T_{\binom{c-1}{2}}]$ respectively, the first Betti numbers of the blowup algebras

$$\dim_{\mathbb{K}} \text{Tor}_1^{S_{\mathcal{R}}}(\mathcal{R}(I), \mathbb{K})_j \quad \text{and} \quad \dim_{\mathbb{K}} \text{Tor}_1^{S_{\mathcal{F}}}(\mathbb{K}[\mathbf{g}], \mathbb{K})_j$$

depend only on the dimension and codimension of $\mathcal{S}_{n_1, \dots, n_d}$ but not on the partition n_1, \dots, n_d . It is not clear whether this holds for higher homological degree.

Remark 5.18 (The Veronese surface). It is interesting to study these questions for all nondegenerate projective varieties $V \subseteq \mathbb{P}^N$ of minimal degree. By the classification of Del Pezzo and Bertini, V is a cone over a smooth such variety, and smooth varieties of minimal degree are precisely the rational normal scrolls, the smooth quadric hypersurfaces, and the Veronese surface $\nu_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$, cf. [16] for details.

We have shown that the Rees ring and the special fiber of any rational normal scroll are Koszul algebras presented by Gröbner bases of quadrics. The same result holds for quadric hypersurfaces, trivially: both blowup algebras are polynomial rings. The ideal $I \subseteq \mathbb{K}[x_0, \dots, x_5]$ of the Veronese surface is of linear type, so the special fiber is a polynomial ring. Surprisingly, it turns out that $\mathcal{R}(I)$ is not a Koszul algebra: a Macaulay2 computation shows that for instance if $\mathbb{Q} \subseteq \mathbb{K}$ then $\dim_{\mathbb{K}} \text{Tor}_6^{\mathcal{R}(I)}(\mathbb{K}, \mathbb{K})_7 = 32$.

This confirms the fact that the Veronese surface exhibits an exceptional behavior in various contexts, cf. [30, Theorem 3.2.5], [10, Example 3.5]. It also gives the first known example of a prime ideal with linear powers [5] whose Rees ring is not Koszul.

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